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Parity violation in four and higher dimensional spacetime with torsion

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Abstract. The possibility of parity violation in a gravitational theory with torsion is extensively explored in four and higher dimensions. In the former case, we have listed our conclusions on when and whether parity ceases to be conserved, with both two- and three-index antisymmetry of the torsion field. In the latter, the bulk spacetime is assumed to have torsion, and the survival of parity violating terms in the four dimensional effective action is studied, using the compactification schemes proposed by Arkani-Hamed–Dimopoulos–Dvali and Randall–Sundrum. An interesting conclusion is that the torsion–axion duality arising in a stringy scenario via the second rank antisymmetric Kalb–Ramond field leads to conservation of parity in the gravity sector in any dimension. However, parity violating interactions do appear for spin-1/2 fermions in such theories, which can have crucial phenomenological implications.

1 Introduction

Torsion in space-time is an interesting possibility in theories of gravitation. In particular, the presence of matter fields with spin has often been suggested as a likely source of torsion. Ever since the Einstein–Cartan (EC) theory was proposed, the customary way to incorporate torsion has been to include it as a tensorial extension to the affine connection, which is antisymmetric in *at least* two indices. It has been further pointed out in some recent studies [1, 2] that, once torsion is present, a similar pseudo-tensorial extension, involving torsion and the completely antisymmetric tensor density, is also possible. This can, in general, cause the violation of parity both in the pure gravity sector (including torsion) and in the coupling of various matter fields with torsion.

In addition, torsion has sometimes been linked with string theories, where it is possible to relate torsion to the rank-2 antisymmetric Kalb–Ramond (KR) field. In such a case, the field strength tensor corresponding to the KR field enters in the connection, and it is antisymmetric in *all three* indices. The constraints imposed by such complete antisymmetry requires a reappraisal of the scenario, especially with regard to parity violation.

The motivation of looking for parity violating gravitational interaction emerges from both theoretical and observational viewpoints. Einstein's general relativity is known to conserve parity. The possibility of a parity violating extension was pointed out in the usual Einstein–Cartan framework by extending the Lagrangian density R, i.e the scalar curvature, by $R + \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$, which is the only possible extension linear in R. Although this new term vanishes identically for Einstein's theory, it yields a non-vanishing parity violating contribution for Einstein–Cartan theory. In [1, 2] it has been pointed out that such a parity violating term comes naturally if one considers a pseudo-tensorial extension of the affine connection. In fact there is no a priori reason to believe that the Cartan extension of the affine connection must have a definite parity i.e. parity conserving only. Thus getting parity violation in this way looks more natural. In addition this allows us to study the coupling of this parity violating term (pseudo-tensorial extension of the affine connection) with other spin fields through the usual minimal coupling prescription [1, 2]. The observational motivation emerges from two important results reported in [3, 4]. In [3] it has been shown that a parity violating gravitational interaction can flip the helicity of a fermion and thereby provides a possible explanation of the well-known neutrino anomaly problem. On the other hand, [4] shows that a parity violating coupling between the electromagnetic and a scalar field can explain the recently observed anisotropy in the cosmic microwave background (CMB) radiation. Indeed the dual scalar of the pseudo-tensor component of the connection discussed above can be identified with such a scalar.

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Further investigations have been recently carried out in the context of theories with large extra dimensions, namely those of Arkani-Hamed–Dimopoulos–Dvali (ADD) [5] and Randall–Sundrum (RS) [6]. In such models torsion exists in the bulk together with gravity, while all the standard model fields are confined to a three-brane. It has been demonstrated [7], that a bulk torsion related to the KR field in ADD type of models has most of its parity violating effects washed out when one compactifies the extra dimensions and considers the projection of bulk fields on the visible brane. In the context of a Randall–Sundrum scenario, very similar conclusions hold in the simplest cases. However, there one reaches the interesting conclusion that in spite of having the same status in the bulk as gravity, effects of the massless mode arising from torsion are heavily suppressed on the standard model brane, thus creating the illusion of a torsionless universe [8].

On the whole, the issue of parity violation in non-torsionfree gravity needs to be addressed in the light of a number of factors, namely
(1) whether the extension due to the torsion field has antisymmetry in *two* or *three* indices;
(2) whether the coupling constants corresponding to the different pseudo-tensorial extensions are the same or different;
(3) the dimensionality of the space in which torsion is assumed to exist;
(4) whether torsion is introduced minimally (i.e. through the terms of least order) or non-linear extensions are to be made if one considers the possibility of parity violation.

In this paper, we present our observations for different cases arising out of combinations of the above possibilities. Although some of the individual points have been discussed earlier in the references given above, an overall perspective is yet to be provided on this unique feature of gravitational interactions. Such a perspective is aimed at in this work.

In Sect. 2, we outline the general features of the mechanism of parity violation induced by torsion. An examination of individual cases in both four and higher dimensions, with the ways in which the parity violating terms can be constructed in each case, is made in Sect. 3. We summarize and conclude in Sect. 4.

2 Torsion and parity violation

2.1 The framework

The connection in EC theory, denoted by $\tilde{\Gamma}^{\mu}_{\nu\lambda}$, is completely asymmetric in all its indices. Upon antisymmetrization of $\tilde{\Gamma}^{\mu}_{\nu\lambda}$ in the two lower indices ν and λ, one obtains a tensor known as "spacetime torsion":

$$H^{\mu}_{\ \nu\lambda} = \frac{1}{2}\left(\tilde{\Gamma}^{\mu}_{\nu\lambda} - \tilde{\Gamma}^{\mu}_{\lambda\nu}\right). \tag{1}$$

Accordingly, $\tilde{\Gamma}^{\mu}_{\nu\lambda}$ can be expressed in terms of the symmetric Christoffel connection $\Gamma^{\mu}_{\nu\lambda}$ and the torsion as follows:

$$\tilde{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - K^{\mu}_{\ \nu\ \lambda}, \tag{2}$$

where $K^{\mu}_{\ \nu\lambda} = H^{\mu}_{\ \nu\lambda} + H_{\lambda\ \nu}^{\ \mu} - H_{\nu\lambda}^{\ \ \ \mu}$ is known as the "contorsion" tensor, constructed out of the torsion tensor in order to preserve the metricity condition in EC theory:

$$\tilde{D}_{\nu}\ g^{\mu\nu} = 0, \tag{3}$$

$\tilde{D}$ being the covariant derivative defined in terms of $\tilde{\Gamma}$. The contorsion tensor is, by construction, antisymmetric in the first and the third covariant (contravariant) indices.

A straightforward way to introduce parity violation through the well-known minimal coupling scheme is to incorporate a pseudo-tensorial extension in the EC connection [1] such that

$$\begin{aligned}\tilde{\Gamma}^{\mu}_{\nu\lambda} \to \tilde{\Gamma}^{\mu}_{\nu\lambda} &= \Gamma^{\mu}_{\nu\lambda} - (H^{\mu}_{\ \nu\lambda} + H_{\lambda\ \nu}^{\ \mu} - H_{\nu\lambda}^{\ \ \ \mu}) \\ &\quad - q\ ({}^{*}H^{\mu}_{\ \nu\lambda} + {}^{*}H_{\lambda\ \nu}^{\ \mu} - {}^{*}H_{\nu\lambda}^{\ \ \ \mu}),\end{aligned} \tag{4}$$

with ${}^{*}H^{\mu}_{\ \nu\lambda}$ having opposite parity properties relative to $H^{\mu}_{\ \nu\lambda}$. The parameter q determines the degree of parity violation, and as a general notation we are using the $*$ for a pseudo-tensor. In general, H and ${}^{*}H$ may be two completely different tensors transforming oppositely under spatial parity, but in that case it is always possible to restore parity through appropriate rephasing of the fields. Therefore, the only situation where one can have a parity violating gravitational field theory is when ${}^{*}H$ is constructed out of H itself and linearly in the completely antisymmetric permutation tensor ϵ. For example, in four dimensions, a valid combination of ϵ and H creating a ${}^{*}H^{\mu}_{\nu\lambda}$ (antisymmetric in ν, λ) may be $\epsilon^{\alpha\beta}_{\ \ \ \nu\lambda} H^{\mu}_{\ \alpha\beta}$, or $\epsilon^{\mu\alpha}_{\ \ \ \beta[\nu} H^{\beta}_{\ \lambda]\alpha}$, as is shown in [1,2].

2.2 H with two-index antisymmetry

As has been mentioned above, H is antisymmetric in two indices in the most general case. If parity has to be violated, then a similar general property has to be attributed to ${}^{*}H$ as well, since the latter is constructed linearly out of the former in a minimal construction. In such a case, the gravitational Lagrangian density, with the surface terms eliminated, turns out to be composed of two parts transforming oppositely under parity. The parity conserving part $\mathcal{L}_{\mathrm{grav}}^{(\mathrm{pc})}$ and the parity violating part $\mathcal{L}_{\mathrm{grav}}^{(\mathrm{pv})}$ are given by

$$\begin{aligned}\mathcal{L}_{\mathrm{grav}}^{(\mathrm{pc})} &= R(g) - H^{\mu}_{\ \nu\lambda}\left(H_{\mu}^{\ \nu\lambda} - 2H^{\nu\lambda}_{\ \ \ \mu}\right) - H^{\alpha}_{\ \alpha\beta} H_{\mu}^{\ \mu\beta} \\ &\quad + O(q^2),\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{grav}}^{(\mathrm{pv})} &= -2q\left(H^{\mu}_{\ \nu\ \lambda}\ {}^{*}H_{\mu}^{\ \nu\lambda} - H^{\mu}_{\ \nu\lambda}\ {}^{*}H^{\nu\lambda}_{\ \ \ \mu}\right. \\ &\quad \left. - {}^{*}H^{\mu}_{\ \nu\lambda}\ H^{\nu\lambda}_{\ \ \ \mu} + 2H^{\alpha}_{\ \alpha\beta}\ {}^{*}H_{\mu}^{\ \mu\beta}\right),\end{aligned} \tag{6}$$

where $O(q^2)$ are the additional parity conserving terms arising in the present scenario; they are of less relevance since we are primarily interested in the terms bearing opposite parity properties relative to the original Cartan terms.

It should also be mentioned here that $\mathcal{L}_{\mathrm{grav}}^{(\mathrm{pv})}$ above is identical to the form proposed in an earlier work [9] where an extra term of the form $\epsilon^{\alpha\beta\mu\nu}\ R_{\alpha\beta\mu\nu}$ was added to the original Einstein–Hilbert Lagrangian. However, the present scheme gives us in addition a guideline for constructing parity violating terms involving matter fields with different spins.

For a spin-1/2 fermion in a spacetime with torsion, the extended Dirac Lagrangian density is given by [10]:

$$\mathcal{L}^{f}_{\mathrm{tor}} = \tag{7}$$
$$\bar{\psi}\ \left[\mathrm{i}\gamma^{\mu}\ \left(\partial_{\mu} - \sigma^{\rho\beta}v^{\nu}_{\rho}g_{\lambda\nu}\partial_{\mu}v^{\lambda}_{\beta} - g_{\alpha\delta}\sigma^{ab}v^{\beta}_{a}v^{\delta}_{b}\tilde{\Gamma}^{\alpha}_{\mu\beta}\right)\right]\ \psi,$$

where v^{μ}_{a} denotes the tetrad connecting the curved space with the corresponding tangent space. The above expression can be decomposed into the terms with opposite parity:

$$\mathcal{L}^{f\ (\mathrm{pc})}_{\mathrm{tor}} = \mathcal{L}^{f}_{\mathrm{E}} \tag{8}$$
$$- \bar{\psi}\left[\mathrm{i}\gamma^{c}\ g_{\alpha\delta}\sigma^{ab}v^{\mu}_{c}v^{\beta}_{a}v^{\delta}_{b}\ \left(H^{\alpha}_{\ \mu\beta} + H_{\beta\ \ \mu}^{\ \alpha} - H_{\mu\beta}^{\ \ \ \alpha}\right)\right]\psi,$$
$$\mathcal{L}^{f\ (\mathrm{pv})}_{\mathrm{tor}} = \tag{9}$$
$$- q\bar{\psi}\left[\mathrm{i}\gamma^{c}\ g_{\alpha\delta}\sigma^{ab}v^{\mu}_{c}v^{\beta}_{a}v^{\delta}_{b}\ \left({}^{*}H^{\alpha}_{\ \mu\beta} + {}^{*}H_{\beta\ \ \mu}^{\ \alpha} - {}^{*}H_{\mu\beta}^{\ \ \ \alpha}\right)\right]\psi,$$

$\mathcal{L}^{f}_{\mathrm{E}}$ being the Dirac Lagrangian density in Einstein gravity. Thus explicit parity violation appears through the term $\mathcal{L}^{f\ (\mathrm{pv})}_{\mathrm{tor}}$ when a spin-1/2 fermion couples to the background torsion. Just a two-index antisymmetry in the torsion tensor is thus sufficient to ensure parity violation in both the pure gravity sector and in the Lagrangian of spin-1/2 particles.

The coupling of torsion with a spin-1 Abelian gauge field A_{μ}, however, runs into problems in maintaining gauge invariance. This is because the corresponding field strength $\tilde{F}_{\mu\nu} = \tilde{D}_{[\mu}A_{\nu]}$ is not invariant under $U(1)$ gauge transformation. This has been a persistent difficulty for torsion with two-index antisymmetry, so long as one wants to remain within the minimal coupling scheme. In a string theoretic scenario, however, this problem can be handled in a manner to be discussed below.

2.3 *H* with three-index antisymmetry

A torsion tensor H with complete antisymmetry in all its indices may be identified with the field strength corresponding to the second rank antisymmetric tensor field $B_{\mu\nu}$ appearing in the massless sector of heterotic string theory. Starting from the Einstein–Cartan action in such an antisymmetric tensor field background one can use the equation of motion for torsion to identify torsion with the KR field strength and trade away the torsion from the action [11].

To cancel the $U(1)$ gauge anomaly and preserve $N = 1$ supersymmetry in the heterotic string theory the field strength $H_{\mu\nu\lambda}$ is augmented suitably with a Chern–Simons (CS) term $A_{[\mu}F_{\nu\lambda]}$ (F being the field strength of a $U(1)$ gauge field A):

$$H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]} + A_{[\mu}\partial_{\nu}A_{\lambda]}. \tag{10}$$

Using this Chern–Simons augmented field strength $H_{\mu\nu\lambda}$ we consider the low energy field theory limit of the bosonic sector of the toroidally compactified string theory. It has been shown [11] that in such a theory a CS term plays the crucial role in resolving the problem of $U(1)$ gauge invariance mentioned above. This is easy to verify since $H_{\mu\nu\lambda}$ as defined above is invariant under the $U(1)$ gauge transformation $\delta A_{\mu} = \partial_{\mu}\omega$ and $\delta B_{\mu\nu} = \omega\ F_{\mu\nu}$ [11].

Now, one can again have a pseudo-tensor ${}^{*}H$ constructed out of the permutation tensor ϵ and H and write in general the torsion as $H_{\mu\nu\lambda} + q\ {}^{*}H_{\mu\nu\lambda}$. The sum as a whole need not be totally antisymmetric, as ${}^{*}H$ can be antisymmetric only in a pair of indices although $H_{\mu\nu\lambda}$ is antisymmetric in all the three indices. Such a construction is explicitly shown in the following section, where we shall also state the specific conditions for retaining parity violating effects in different sectors. Due to the presence of the CS term, the Einstein–Cartan–Kalb–Ramond (ECKR) Lagrangian density involves the gauge field A. Therefore, following the formalism in [11] we can express the Lagrangian density for the ECKR–gauge field coupling as

$$\mathcal{L}^{\mathrm{gauge}}_{\mathrm{ECKR}} = R(g) - \frac{1}{12}(H^{\mu\nu\lambda} + q\ {}^{*}H^{\mu\nu\lambda})(H_{\mu\nu\lambda} + q\ {}^{*}H_{\mu\nu\lambda})$$
$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{11}$$

The Lagrangian density for the ECKR–fermion coupling is given by

$$\mathcal{L}^{f}_{\mathrm{ECKR}} = \mathcal{L}^{f}_{\mathrm{E}} - \bar{\psi}\left[\mathrm{i}\gamma^{c}g_{\alpha\delta}\sigma^{ab}v^{\mu}_{c}v^{\beta}_{a}v^{\delta}_{b} \right. \tag{12}$$
$$\left. \times \left\{H^{\alpha}_{\ \mu\beta} + q\left({}^{*}H^{\alpha}_{\ \mu\beta} + {}^{*}H_{\beta\ \ \mu}^{\ \alpha} - {}^{*}H_{\mu\beta}^{\ \ \ \alpha}\right)\right\}\right]\psi.$$

3 Construction of **H* in different dimensions

3.1 The general outlook

So far we have discussed in a general way the possibility of parity violation arising from ${}^{*}H$. Now we shall concentrate on various ways of constructing ${}^{*}H$ out of H in different spacetime dimensions.

Depending on the dimensionality, ${}^{*}H$ can be constructed using linear as well as higher powers of H. In particular, it is straightforward to see that
(a) in even spacetime dimensions (4, 6 ...) ${}^{*}H$ must be constructed using an odd number of the H, i.e., ${}^{*}H$ is linear, cubic, ... in the H;
(b) in odd spacetime dimensions (5, 7, ...) ${}^{*}H$ must contain an even number of the H and therefore can be bilinear, quadrilinear, ... in H.

Note that since the three-form H is equal to $\mathrm{d}B + A \wedge F$, dimensional arguments tell us that an ${}^{*}H$ constructed out of higher powers of the H gives parity violating effects suppressed by correspondingly higher powers of the Planck mass.

3.2 Construction of **H* in four dimensions

3.2.1 Minimal construction

We now consider an H which is totally antisymmetric in all three indices. In four dimensions, a minimally constructed

${}^*H$ consists of terms linear in H, whence the pseudo-tensorial connection can generically be written as [2]

$$q\ {}^*H^{\mu}_{\ \nu\lambda} = q_1\ \epsilon^{\alpha\beta}_{\ \ \nu\lambda}\ H^{\mu}_{\ \alpha\beta} + q_2\ \epsilon^{\mu\sigma}_{\ \ \rho[\nu}\ H^{\rho}_{\ \lambda]\sigma}. \tag{13}$$

However, if the coupling strengths q_1 and q_2 are equal (which is the situation corresponding to complete anti-symmetry in the pseudo-connection), the above expression vanishes identically as a whole. This can be verified easily on observing that one can always replace the totally antisymmetric three-tensor $H_{\mu\nu\lambda}$ – a three-form – by its *Hodge*-dual one-form, i.e., a pseudo-vector h_σ, as follows:

$$H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\sigma}\ h^\sigma. \tag{14}$$

Therefore q_1 and q_2 must always differ, which implies that we are left with the case where the term ${}^*H^{\mu}_{\ \nu\lambda}$ is antisymmetric in ν and λ only. This is not an unnatural assumption, since there is no symmetry of the theory demanding the equality of the two charges.

Even with q_1 and q_2 unequal, a rather interesting thing is observed. If one considers the parity violating part of the gravity sector (see (6)) and uses the above duality relation, it is straightforward to see that $\mathcal{L}^{(\mathrm{pv})}_{\mathrm{grav}} = 0$ identically. Thus a Kalb–Ramond type of torsion cannot violate parity in the effective scalar curvature.

A similar conclusion follows for Abelian gauge fields, too. The gauge-invariant ECKR-Lagrangian density along with the gauge field A (see (11)) can now be separated into parity conserving (pc) and parity violating (pv) parts as follows:

$$\begin{aligned}\mathcal{L}^{\mathrm{gauge\ (pc)}}_{\mathrm{ECKR}} &= R(g) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{12}\left(\partial_{[\mu}B_{\nu\lambda]} + A_{[\mu}F_{\nu\lambda]}\right)\left(\partial^{[\mu}B^{\nu\lambda]} + A^{[\mu}F^{\nu\lambda]}\right) \\ &+ O(q_1,q_2)^2, \end{aligned}\tag{15}$$

$$\begin{aligned}\mathcal{L}^{\mathrm{gauge\ (pv)}}_{\mathrm{ECKR}} &= -\frac{1}{6}\ (q_1 + 2q_2) \\ &\times \epsilon^{\rho\alpha}_{\ \ \beta\sigma}\ \left(\partial_{[\lambda}B_{\rho\alpha]} + A_{[\lambda}F_{\rho\alpha]}\right)\left(\partial^{[\lambda}B^{\beta\sigma]} + A^{[\lambda}F^{\beta\sigma]}\right).\end{aligned}\tag{16}$$

However, once again the relation (14) can be employed to check that the parity violating term $\mathcal{L}^{\mathrm{gauge\ (pv)}}_{\mathrm{ECKR}}$ vanishes identically. This is because all the terms, including those from the Chern–Simons extension, are three index antisymmetric and therefore dual to a vector in four dimension by the relation (14). So our conclusion is that the theory is parity conserving in both the gravity and electromagnetic sector even for $q_1 \neq q_2$.

It is worth mentioning here that in a recent work [12], it has been argued that there can be an alternative way of incorporating parity violation in the coupling of the gauge field with torsion by extending the Chern–Simons term in the modified field strength $H_{\mu\nu\lambda}$ by the dual of the Maxwell field tensor $F^{\mu\nu}$. Such a term generates a parity violating interaction between the gauge field and the torsion.

Once the torsion tensor is identified with the KR field, the pseudo-tensorial extension of the connection makes the KR coupling to a spin-1/2 fermion parity violating. In terms of the *axion* that appears in the string spectrum and that is defined through the duality relation

$$\partial_{[\mu}B_{\nu\lambda]} = \epsilon_{\mu\nu\lambda\sigma}\ \partial^\sigma\ \phi, \tag{17}$$

the Lagrangian density in the fermionic sector is given by

$$\begin{aligned}\mathcal{L}^f_{\mathrm{ECKR}} &= \mathcal{L}^f_{\mathrm{E}} \\ &+ 8\ (q_1 - q_2)\ \bar\psi\ \left(\mathrm{i}\gamma_c\ \sigma^{ab}\ v^\lambda_a v^\mu_b v^\nu_c\ g_{\nu\lambda}\ \partial_\mu\ \phi\right)\ \psi \\ &+ \bar\psi\ \Big(\mathrm{i}\gamma_c\ \sigma^{ab}\ v^\lambda_a v^\mu_b v^\nu_c \\ &\times\ \left[2q_1\ \epsilon^{\alpha\beta}_{\ \ \nu\lambda}\ A_{[\alpha}F_{\beta\mu]} - (q_1 - 2q_2)\ \epsilon^{\alpha\beta}_{\ \ \mu\nu}\ A_{[\alpha}F_{\beta\lambda]}\right]\Big)\ \psi.\end{aligned}\tag{18}$$

This Lagrangian density is manifestly parity violating, through both the axion ϕ and the CS term. Thus fermionic interactions constitute the benchmark of parity violation induced by torsion of the above kind, albeit with the special requirement $q_1 \neq q_2$. Moreover, due to the presence of the CS term in the connection, an interaction between the gauge field and the fermion appears. Though the term is suppressed by two powers of the Planck mass, such an interaction may be interesting for future studies.

Before we move on to the next topics, we summarize below our main conclusions on parity violation with torsion in four dimensions, with the pseudo-tensorial extension always kept linear in H.

(1) When the torsion tensor is antisymmetric *only in the two lower indices* (i.e. in a model-independent representation of torsion), parity violation is always observed both in the pure gravity sector (i.e. in the effective scalar curvature) and in the coupling of matter fields to torsion. However, the coupling of torsion to massless gauge fields is difficult to ensure unless one goes beyond the minimal scenario.

(2) When the torsion tensor is antisymmetric in *all three indices* (i.e. one can write it both as the strength of the antisymmetric Kalb–Ramond tensor field and as the dual of a pseudoscalar field), the pseudo-tensorial extension ${}^*H$ identically vanishes so long as it is also constructed as *antisymmetric in all three indices*. Thus there is no possibility of parity violation in such a case.

(3) When the torsion tensor is antisymmetric in *all three indices*, it is still possible to have only a *two-index antisymmetry* in the pseudo-tensorial part ${}^*H$, by imposing inequality of the two couplings q_1 and q_2. In such a case, too, the gravity sector and the gauge field sector still turn out to be parity conserving. However, spin-1/2 fields have a parity violating coupling with torsion in such a case.

3.2.2 Non-minimal construction

We have already seen that no parity violation occurs in the gauge and gravity sectors for the minimal (linear) extension in four dimensions. Here we look for the possibility of parity violation in these sectors with the leading non-minimal terms in the extension. As mentioned earlier, in four dimensions, a pseudo-tensor ${}^*H$ constructed using H

can, in general, have terms containing only odd powers of H. Therefore, the most general pseudo-tensorial part of the affine connection can be schematically expressed as

$$^{*}H = \epsilon H + \epsilon\ HHH + \epsilon\ HHHHH + \ldots \qquad (19)$$

The set of possible terms trilinear in R (suppressing the charges multiplying the various terms) in the above expression is given by

$$\epsilon^{\alpha\beta}{}_{\nu\lambda}\ H^{\mu}{}_{\alpha\rho}\ H^{\sigma\kappa}{}_{\beta}\ H^{\rho}{}_{\sigma\kappa} + \epsilon^{\alpha\beta}{}_{\rho[\nu}\ H^{\sigma}{}_{\lambda]\kappa}\ H^{\rho\kappa}{}_{\sigma}\ H^{\mu}{}_{\alpha\beta}$$
$$+ \epsilon^{\alpha\mu}{}_{\beta\rho}\ H^{\beta\sigma}{}_{[\nu}\ H^{\kappa}{}_{\lambda]\alpha}\ H^{\rho}{}_{\sigma\kappa}$$
$$+ \epsilon^{\alpha\mu}{}_{\beta\rho}\ H^{\beta}{}_{\nu\lambda}\ H^{\rho\sigma}{}_{\kappa}\ H^{\kappa}{}_{\alpha\sigma} + \text{similar terms.} \qquad (20)$$

In the special case of a completely antisymmetric pseudo-tensorial connection, there are a number of allowed terms for each non-minimal order construction. However, similar to the minimal construction case, terms of each variety in the non-minimal construction can be shown to vanish on using the general relation (14). Thus we can make the following generic statement: *it is, in no way, possible in a four dimensional Lagrangian to have a pseudoscalar term built out of completely antisymmetric three-tensors raised to any index.*

When the field H is only two-index antisymmetric then the non-minimal extensions no longer vanish. However, it can be explicitly checked that no parity violating term in the Lagrangian density in four dimensions appears either in the gravity sector or in the coupling with the gauge fields up to a term trilinear in H in the connection. In the fermionic sector, parity violating terms from the non-minimal extensions do appear in the Lagrangian density. However, such terms are hardly of any significance as they are suppressed by increasingly higher powers of the Planck mass.

3.3 Construction of $^{*}H$ in five dimensions

Considering that torsion (or, equivalently, the KR field) coexists alongside gravity in the bulk, we find that in five dimensional spacetime the pseudo-tensor $^{*}H$ constructed from H has to be at least bilinear in the latter. The most general pseudo-tensorial part of the affine connection, antisymmetric in a pair of indices, can now be written as

$$^{*}H^{\mu'}{}_{\nu'\lambda'}$$
$$= \left(q_1\ \epsilon_{\alpha'\beta'\gamma'\nu'\lambda'} H^{\mu'\alpha'\delta'} + q_2\ \epsilon^{\mu'}{}_{\alpha'\beta'\gamma'[\nu'} H_{\lambda']}{}^{\alpha'\delta'}\right) H^{\beta'\gamma'}{}_{\delta'}$$
$$+ \left(q_3\ \epsilon^{\mu'}{}_{\alpha'\beta'\gamma'\delta'} H_{\nu'\lambda'}{}^{\alpha'} + q_4\ \epsilon_{\alpha'\beta'\gamma'\delta'[\nu'} H_{\lambda']}{}^{\mu'\alpha'}\right) H^{\beta'\gamma'\delta'}$$
$$+\ q_5\ \epsilon_{\alpha'\beta'\gamma'\delta'[\nu'} H_{\lambda']}{}^{\alpha'\beta'} H^{\mu'\gamma'\delta'}, \qquad (21)$$

where the primed indices $\mu', \nu', \ldots$, etc. run all over both the usual four-dimensional spacetime and the extra space dimension y. The coupling strengths q_1, q_2, q_3, q_4 and q_5 are, in general, different from each other, thereby leaving $^{*}H$ to be antisymmetric in two indices. The special case of a totally antisymmetric pseudo-tensor $^{*}H$ can be encountered if we set $q_1 = -q_2,\ q_3 = q_4$ and put $q_5 = 0$. Unlike in four dimensions, here the totally antisymmetric $^{*}H$ gives a non-vanishing contribution to the connection.

With this modified connection in five dimensions, we now examine the parity violating effect in the effective four dimensional theory with two compactification mechanisms, viz., Arkani-Hamed–Dimopoulos–Dvali (ADD) [5] and Randall–Sundrum (RS) [6]. We compute the parity violating part of the four dimensional Lagrangian density for these two compactification schemes. For the sake of convenience we consider here only the terms multiplying the q_1 and q_2 terms of (21), with $q_1 \neq q_2$ in general. The conclusions are, however, not affected by this simplification.

3.3.1 Compactification in ADD scenario

Although the ADD type of models are phenomenologically disfavored in five dimensions, we include it here for completeness. In such a model [5], the compact and Lorenz degrees of freedom can be factorized and the string scale M_{s} (expected to lie between a few TeV and a few tens of TeV) controls the strength of gravity in $(4+n)$ dimensions. M_{s} is related to the four dimensional Planck scale M_{P} by $M_{\mathrm{s}}^{n+2}/M_{\mathrm{P}}^2 \sim R^{-n}$, R being the compactification radius. Compactification of the n extra dimensions leads to a tower of Kaluza–Klein (KK) modes on a visible three-brane and as such a massless field in the bulk gives rise to a massive spectrum $m_{\mathbf{n}}^2 = 4\pi^2\mathbf{n}^2/R^2$ with $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ [13]. In a physical process, the summation over these towers of fields, convoluted with the corresponding density of states, causes an enhancement, despite an M_{P}-suppression of individual coupling. Thus "new physics" is found to intervene at the TeV scale, thereby providing a natural cut-off to the electroweak theory.

For a bulk KR field $B_{\mu'\nu'}$, the ADD compactification in general gives rise to a set of tensor fields $B^{\mathbf{n}}_{\mu\nu}$, vector fields $B^{\mathbf{n}}_{\mu}$ and scalar fields $\chi^{\mathbf{n}}$ in a four dimensional effective theory. However, one can assume the bulk $B_{\mu'\nu'}$ to be block-diagonal in compact and non-compact dimensions [7], i.e., $B^{\mathbf{n}}_{\mu}$ can be taken to be zero without any loss of generality. Now, following the standard toroidal compactification procedure shown in [13], we obtain the four dimensional effective parity violating part of the Lagrangian density for KR–fermion coupling:

$$\mathcal{L}_f^{(\mathrm{pv})}$$
$$= 2q_1\ \bar{\psi}\ \left[\sum_{n,n',m,m'} \mathrm{i}\gamma_c\sigma_{ab}v^a_\mu\epsilon^{\alpha\beta bc}\ \left\{\frac{2\pi i}{R}\ n\ g^{\mu\rho(m)}\ g^{\nu\sigma(m')}\right.\right.$$
$$\times \left(B^{(n)}_{\nu\rho}\ \partial_{[\alpha}B^{(n')}_{\beta\sigma]} + 2B^{(n)}_{\sigma\beta}\ \partial_{[\rho}B^{(n')}_{\alpha\nu]}\right)$$
$$-\frac{4\pi^2}{R^2}\ nn'\ g^{\mu\rho(m)}\ \zeta^{\nu(m')}$$

$$\times \left(B^{(n)}_{\nu\rho}\ B^{(n')}_{\alpha\beta} + 2B^{(n)}_{\rho\alpha}\ B^{(n')}_{\sigma\beta}\right)\Big\}\Bigg]\ \psi$$

$$+ \frac{q_1+2q_2}{2}\ \bar{\psi}\ \left[\sum_{n,n',m} \gamma^c\sigma^{ab}\gamma^5 v^\alpha_a v^\beta_b v^\lambda_c \left\{\frac{2\pi \mathrm{i}}{R}\ n\ g^{\nu\sigma(m)}\right.\right.$$

$$\times \left(B^{(n)}_{\nu\lambda}\ \partial_{[\alpha}B^{(n')}_{\beta\sigma]} + 2B^{(n)}_{\sigma\beta}\ \partial_{[\lambda}B^{(n')}_{\alpha\nu]}\right)$$

$$\left.\left. - \frac{4\pi^2}{R^2}\ nn'\ \zeta^{\sigma(m)} \left(B^{(n)}_{\sigma\lambda}\ B^{(n')}_{\alpha\beta} + 2B^{(n)}_{\lambda\alpha}\ B^{(n')}_{\sigma\beta}\right)\right\}\right]\ \psi, \quad (22)$$

where $\zeta^{\sigma(n)} = g^{\sigma y(n)}$ (where y stands for the extra dimensions), and with the contributions due to the CS terms which are suppressed by higher powers of Planck mass in the above expression.

3.3.2 Compactification in the RS scenario

In the RS framework, we have a non-factorizable geometry and as such the metric contains a so-called "warp" factor which is an exponential function of the compact space dimension y:

$$\mathrm{d}s^2 = \mathrm{e}^{-2kr_\mathrm{c}|y|}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - r_\mathrm{c}^2\mathrm{d}y^2, \qquad (23)$$

where r_c is the compactification radius on a Z_2 orbifold, and $k \sim M_5$, the higher dimensional Planck mass. For the bulk KR field $B_{\mu'\nu'}$ in this scenario, one can use the standard decomposition technique used, for example, in [14]:

$$B_{\mu'\nu'}(x,y) = \sum_n \frac{B^{(n)}_{\mu\nu}(x)}{\sqrt{r_\mathrm{c}}}\ \xi^{(n)}(y), \qquad (24)$$

which, on the visible brane, is given by

$$B_{\mu\nu}(x) = \sum_n \frac{B^{(n)}_{\mu\nu}(x)}{\sqrt{r_\mathrm{c}}}\ \xi^{(n)}(\pi). \qquad (25)$$

The couplings are controlled by appropriate warp factors arising from ξ [14]. The four dimensional effective parity violating part of the Lagrangian density for KR–fermion coupling in this case is given by

$$\mathcal{L}_f^{\,(\mathrm{pv})} = -\frac{2q_1}{r_\mathrm{c}^3}\ \bar{\psi}\ \left[\mathrm{i}\gamma^c\sigma^{ab}\ \mathrm{e}^{6\pi kr_\mathrm{c}}\ \eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\delta\kappa}\ \epsilon_{\alpha\beta bc}\right.$$

$$\times \sum_{n,n'}\Big\{B^{(n)}_{\delta a}\partial_{[\rho}B^{(n')}_{\sigma\kappa]}\ \xi'^{(n)}(\pi)\xi^{(n')}(\pi)$$

$$\left. +\partial_{[a}B^{(n)}_{\rho\delta]}B^{(n')}_{\kappa\sigma}\ \xi^{(n)}(\pi)\xi'^{(n')}(\pi)\Big\}\right]\psi$$

$$-\frac{q_1+2q_2}{2r_\mathrm{c}^3}\ \bar{\psi}\Big[\gamma^c\sigma_{ab}\gamma_5\ \mathrm{e}^{6\pi kr_\mathrm{c}}\ \eta^{a\rho}\eta^{b\sigma}\eta^{\delta\kappa}$$

$$\times \sum_{n,n'}\Big\{B^{(n)}_{\delta c}\partial_{[\rho}B^{(n')}_{\sigma\kappa]}\ \xi'^{(n)}(\pi)\xi^{(n')}(\pi)$$

$$+\partial_{[c}B^{(n)}_{\rho\delta]}B^{(n')}_{\kappa\sigma}\ \xi^{(n)}(\pi)\xi'^{(n')}(\pi)\Big\}\Big]\psi. \qquad (26)$$

where $\xi'^{(n)}(\pi) = \mathrm{d}\xi^{(n)}/\mathrm{d}y\ |_{(y=\pi)}$.

It should be mentioned in this context that a five dimensional scenario also admits of an additional term of the form

$$\mathcal{L}^{\mathrm{pv}}_{\mathrm{HB}} = M_\mathrm{P}\ \epsilon^{\mu\nu\lambda\alpha\beta}H_{\mu\nu\lambda}B_{\alpha\beta}. \qquad (27)$$

Such a term is invariant under the Kalb–Ramond gauge transformation $\delta B_{\mu\nu} = \partial_{[\mu}\omega_{\nu]}$, modulo a divergence term. However, it is not invariant under the $U(1)$ gauge transformation of the KR field, which we have introduced to justify the Chern–Simons terms defined earlier. Therefore, a term of this form survives only if torsion does not couple to electromagnetism, at least through a Chern–Simons term.

Once a term of this kind exists, one hopes to generate a parity violation in four dimensions when the fifth dimension is compactified à la Randall–Sundrum. However, it is found that the presence of such a term makes the $B_{\mu\nu}$ field selfdual or anti-selfdual and the resulting four dimensional action conserves parity. We shall report the details of such a scenario in a forthcoming paper.

3.4 Construction of $^*H$ in six dimensions

The construction of $^*H$ in six dimensions can only be completely antisymmetric in all covariant (contravariant) indices [7]: $^*H_{\mu'\nu'\lambda'} = \epsilon^{\alpha'\beta'\gamma'}{}_{\mu'\nu'\lambda'}H_{\alpha'\beta'\gamma'}$. As has been the cases in four and five dimensions, if one calculates here the parity violating part of the EC–KR–Maxwell Lagrangian, i.e., the term $^*H^{\mu'\nu'\lambda'}H_{\mu'\nu'\lambda'}$, it turns out to be zero again. Moreover, for the KR–fermion coupling, it has been shown explicitly in [7] that, although the augmentation of the covariant derivative with the pseudo-tensorial part in the presence of torsion causes parity violation in the bulk, the ensuing theory in four dimensions turns out to be parity conserving. This can be understood from the fact that, upon an ADD type compactification, one can obtain the following KR coupling to the spin-1/2 fermion of mass m:

$$\mathcal{L}_f = \mathcal{L}_f^{\mathrm{E}} + M_\mathrm{P}^{-1}\ \sum_{\mathbf{n}} \bar{\psi}\ \left(\mathrm{i}\gamma^\mu\sigma^{\nu\lambda}\ \partial_{[\mu}B^{\mathbf{n}}_{\nu\lambda]}\right)\ \psi$$

$$-\frac{144qm}{M_\mathrm{P}}\ \bar{\psi}\ \mathrm{i}\gamma_5\chi\ \psi, \qquad (28)$$

where $\mathcal{L}_f^{\mathrm{E}}$ is the four dimensional Dirac Lagrangian in Einstein gravity, q being the charge of the pseudo-connection and χ the scalar field in the KK spectrum for $B_{\mu'\nu'}$. From the viewpoint of a parity transformation in four dimensions, this Lagrangian is invariant, since we can always use the phase freedom of the fields $B^{\mathbf{n}}_{\mu\nu}$ and χ independently on the three-brane. It has also been argued in [7] that the above feature of getting no parity violation in any sector in six dimensions holds in a RS framework as well.

4 Summary and conclusions

We have made a general survey of the role of spacetime torsion as a possible source of parity violation, evinced from its interaction with both curvature and various spin fields. We have shown that while a completely antisymmetric torsion (originating from a Kalb–Ramond field in a string inspired model) can induce parity violation only in the spin-$1/2$ fermion sector but not in the curvature or $U(1)$ gauge sector. A two-index antisymmetric torsion can however violate parity in all spin sectors.

We have also generalized these results into higher spacetime dimensions. These results are specially significant in studying parity violation in phenomenological models originating from D-branes. Postulating the existence of torsion (identified with the KR field) in the bulk in each case, we still find that parity is always restored when one considers its coupling to curvature. On the other hand, the fermionic sector is seen to violate parity in the resulting four dimensional theory obtained upon compactification of the extra dimensions à la Randall–Sundrum and ADD. In each of the above cases, all the parity violating couplings are explicitly calculable. These parity violating couplings may turn out to be phenomenologically significant in the context of solar neutrino problem [3].

We conclude with the observation that the curvature (or gravity) sector and electromagnetic sector are always seen to be shielded from parity violating effects whenever the torsion tensor is fully antisymmetric in all three indices. This, in turn, is traced to the fact that such a tensor can always be expressed in terms of its dual axion field. Thus parity conservation in gravity, space-time torsion notwithstanding, has a rather striking relationship with duality. We have thus exhaustively described the possibilities of generating parity violating interactions through spacetime torsion with special emphasis on string inspired models. Various phenomenological implications of the results presented in this work may now be investigated for $(3+1)$ dimensional as well as higher dimensional theories. It may be noted from (4) that q measures the relative strength between the parity conserving and parity violating part in the Cartan extension of the affine connection. Thus to determine q one must look into phenomena originating from both the parity violating and parity conserving part. Calculating the helicity flip amplitude from left handed to right handed neutrino and the resulting change of flux of the incoming left handed solar neutrino [3] we can compare this to the experimental data to estimate the parity violating component. The parity conserving part does not contribute in this process. Data from CMB anisotropy can also be used to determine the parity violating part [15]. Moreover the experimental value of the optical rotation of the plane of polarization of the distant galactic polarized radiations, over and above the usual Faraday rotation, may be used to determine both the parity conserving as well as parity violating components [16]. For the higher dimensional theories like ADD, the scenario (22) indicates that only the massive Kaluza–Klein tower of the KR field contributes in the KR–fermion interaction term whereas in the RS scenario, (26) implies that both the massless as well as the massive modes of the KR field interact with the fermions. As the massless mode in the RS scenario is shown to be suppressed by the large warp factor on the visible brane [8], the massive KR modes in these higher dimensional theories are expected to play crucial roles in the forthcoming TeV scale experiments.

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